Optimal Flight-Path-Angle Transitions in Minimum-Time Airplane Climbs

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If dependence of drag D on lift L is suppressed by calculating the induced drag corresponding to L = weight W, the minimum-time climb path, obtained either by the energy state analysis or by Green's Theorem, leads, as is well known, to discontinuities in the flight-path angle γ. This requires, of course, an unreasonable increase in lift, positive or negative! If the reciprocal of maximum L/D is treated as a small parameter ϵ, dependent on Mach number, a more complete analysis reveals that the discontinuities in γ are replaced by transitional "boundary layers" on time scales of order $\epsilon^{1/2}$, during which L/D is of order $\epsilon^{-1/2}$ rather than ϵ^{-1}.

I. Introduction

If 1) the gravity field is treated as uniform, 2) decrease in airplane mass is neglected, and 3) the dependence of drag D on lift L is suppressed by either neglecting the induced drag or calculating it as if lift exactly balances weight, then the flight-path-angle γ (or its sine) may be regarded as the control; in addition (see Ref. 1), the optimal paths in the $h-V$ plane are made up of singular arcs ($|\sin\gamma|<1$) and of tributrary arcs ($\gamma=\pm\pi/2$), as illustrated in Figs. 1.

The singular arc, or arcs, can also be obtained (see Ref. 2) by regarding V as a control and maximizing the time rate of increase of the single (energy) state $E=V^2/2+gh$, which rate is seen easily to be independent of γ.

The discontinuity in γ at a junction of a tributary with a singular arc (and of arrival and departure from the state-constrained arc $h=0$) is due to neglect of physical limits on lift magnitude, since this approximate theory fails to account for a sharp rise in induced drag when maximum lift is employed.

Ardema[3] has carried out a "boundary-layer" analysis, which starts by examining the optimal pattern of changes in h and γ while the energy state E remains constant. His numerical results have been encouraging, in spite of the fact that the boundary-layer time scale is not small in comparison with that associated with energy change.

This paper will attempt a different boundary-layer analysis in which γ will be assumed to vary more rapidly than V and h. The *optimal* behavior of γ near the junctions in Figs. 1 will, indeed, involve relatively rapid changes in γ, at least if the airplane has high L/D capability.

II. Analysis of Transitions to and from a Singular Arc

The equations of motion are

$$\dot{h}=V\sin\gamma \tag{1a}$$

$$\dot{V}=(T-D)/m-g\sin\gamma \tag{1b}$$

$$V\dot{\gamma}=L/m-g\cos\gamma \tag{1c}$$

Drag D will be assumed to obey the parabolic formula

$$D(h,V,L)=D_0(h,V)+[\epsilon^2L^2/D_0(h,V)] \tag{2}$$

where

$$\epsilon=\frac{1}{2(L/D)_{\max}}$$

ϵ being in general dependent on Mach number. Variation in mass m still will be neglected, but thrust T can vary with h and V.

An appropriate Hamiltonian is

$$\mathcal{H}=-1+\lambda_h V\sin\gamma+\lambda_V\left(\frac{T-D_0-(\epsilon^2L^2/D_0)}{m}-g\sin\gamma\right)+\frac{\lambda_\gamma}{V}\left(\frac{L}{m}-g\cos\gamma\right) \tag{3}$$

and the optimal control (the lift) is given by maximizing $\mathcal{H}$ as

$$L=\lambda_\gamma D_0/2\epsilon^2 V\lambda_V \tag{4}$$

The adjoint rates are

$$\dot{\lambda}_h=\frac{\lambda_V}{m}\frac{\partial}{\partial h}\left(D_0-T+\frac{\epsilon^2L^2}{D_0}\right) \tag{5a}$$

$$\dot{\lambda}_V=-\lambda_h\sin\gamma+\frac{\lambda_\gamma}{V^2}\left(g\cos\gamma-\frac{L}{m}\right)+\frac{\lambda_V}{m}\frac{\partial}{\partial V}\left(D_0-T+\frac{\epsilon^2L^2}{D_0}\right) \tag{5b}$$

$$\dot{\lambda}_\gamma=-\lambda_\gamma(g/V)\sin\gamma-S\cos\gamma \tag{5c}$$

where

$$S=V\lambda_h-g\lambda_V \tag{6}$$

which is the switch-function for the "control" $\sin\gamma$ in the case, $\epsilon=0$, when induced drag is neglected, in which case $\lambda_\gamma=0$.

Now $\dot{S}$ can be expressed in the form

$$\dot{S}=-\frac{g\lambda_V}{mV}F(h,V)+\frac{S}{mV}(T-D_0)-\frac{g\lambda_\gamma}{V^2}\left(g\cos\gamma-\frac{L}{m}\right)+(\text{terms with }\epsilon^2) \tag{7}$$

where

$$F(h,V)\equiv\left(1+V\frac{\partial}{\partial V}-\frac{V^2}{g}\frac{\partial}{\partial h}\right)(D_0-T) \tag{8}$$

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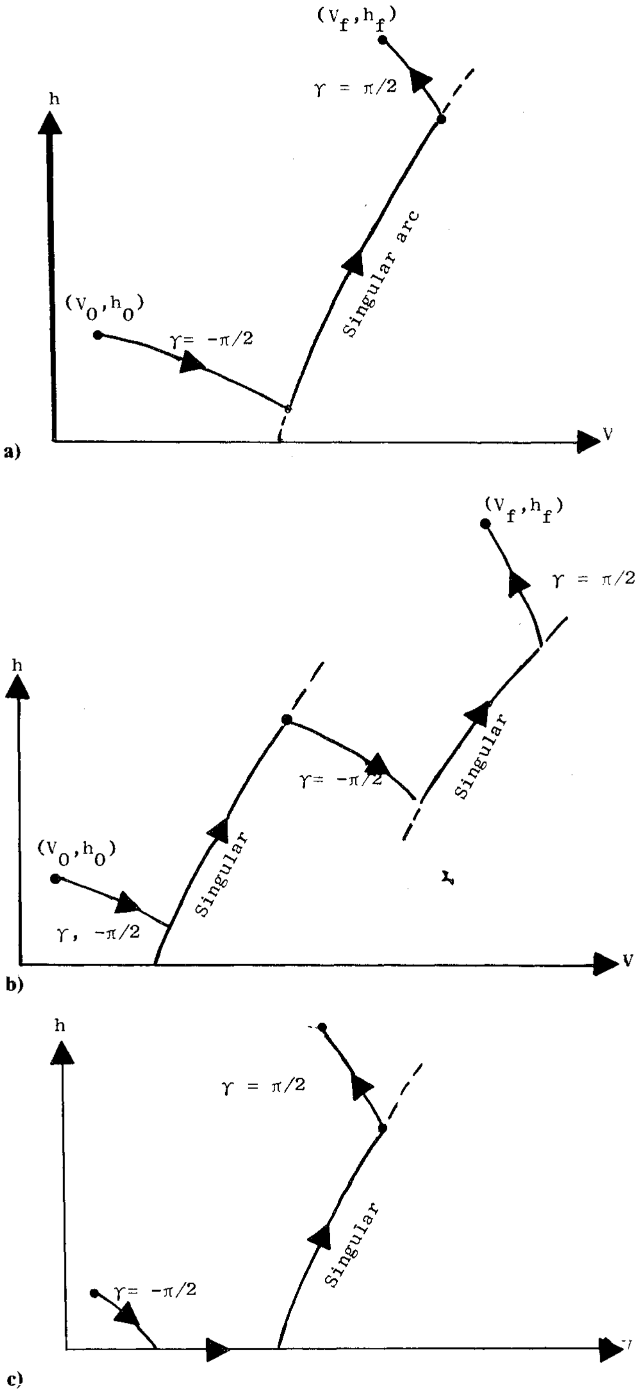


Fig. 1 Cornered path in $V-h$ plane.

The singular arc in Figs. 1, on which $|\sin\gamma| < 1$ so that $S=0$, is thus given by $F(h,V)=0$.

Furthermore, $\ddot{S}$ is expressible as

$$\ddot{S} = -\frac{g\lambda_V}{mV}\left[\frac{d}{dt}F(h,V)\right] + (\text{terms with } F,S,\dot{S},\lambda_\gamma,\epsilon^2) \qquad (9)$$

and

$$\left[\frac{d}{dt}F(h,V)\right]_{\epsilon=0} = -\sin\gamma(gF_V - VF_h) + \frac{T-D_0}{m}F_V \qquad (10)$$

The vanishing of F on the singular arc thus implies that

$$\left[\frac{d}{dt}F(h,V)\right]_{\epsilon=0} = (\sin\gamma_S - \sin\gamma)(gF_V - VF_h) \qquad (11)$$

where γ_S is the flight-path-angle along the singular arc. The vanishing of the Hamiltonian $\mathcal{H}$, together with the vanishing of S along the singular arc, shows that $\lambda_V = m/(T-D_0)$ on the singular arc (for $\epsilon=0$), and the dominant terms in $\dot{S}$, $\ddot{S}$, for $\epsilon \neq 0$, are thus

$$\dot{S} \cong -gF/V(T-D_0) \qquad (12a)$$

$$\ddot{S} \cong k_1(\sin\gamma - \sin\gamma_S) \qquad (12b)$$

where

$$k_1 = k_1(h,V) = \frac{g}{V(T-D_0)}(gF_V - VF_h) \qquad (13)$$

But, assuming that $|L| \gg mg$ for the main part of a transition to or from the singular arc,

$$\dot{\gamma} \cong L/mV = \lambda_\gamma D_0 / 2\epsilon^2 m V^2 \lambda_V$$

i.e.,

$$\dot{\gamma} \cong k_2 \lambda_\gamma / \epsilon^2 \qquad (14)$$

where

$$k_2 = k_2(h,V) = D_0(T-D_0)/2m^2V^2 \qquad (15)$$

so that

$$\ddot{\gamma} = -(k_2/\epsilon^2)S\cos\gamma + (\text{terms with } \dot{\gamma}) \qquad (16)$$

We now introduce the rescaled variables

$$\tau = (t/\sqrt{\epsilon})(k_1 k_2)^{1/4} \qquad (17a)$$

$$\sigma = (S/\epsilon)(k_2/k_1)^{1/2} \qquad (17b)$$

and obtain

$$\frac{d^2\gamma}{d\tau^2} = -\sigma\cos\gamma + O(\epsilon^{1/2}) \qquad (18a)$$

$$\frac{d^2\sigma}{d\tau^2} = \sin\gamma - \sin\gamma_S + O(\epsilon^{1/2}) \qquad (18b)$$

Near $\gamma = \gamma_S$, the transition equations (18) reduce to

$$\frac{d^4(\gamma-\gamma_S)}{d\tau^4} \cong -(\gamma-\gamma_S)\cos^2\gamma_S \qquad (19)$$

and near $\gamma = \pm\pi/2$, where $|\tau|$ becomes large,

$$\sigma \sim \pm(\tau^2/2)(1 \mp \sin\gamma_S)$$

so that

$$\frac{\pi}{2} \mp \gamma \sim Ae\exp[(\tau^2/2\sqrt{2})\sqrt{1\mp\sin\gamma_S}]$$
$$+ Be\exp[-(\tau^2/2\sqrt{2})\sqrt{1\mp\sin\gamma_S}] \qquad (20)$$

which can only approach zero for large $|\tau|$ if $A=0$.

The required solution of Eqs. (18), e.g., for the transition from the singular arc to the final arc $\gamma = \pi/2$ in Fig. 1a, is that solution for which $\gamma \to \gamma_S$ as $\tau \to -\infty$ and $\gamma \to \pi/2$ as $\tau \to +\infty$. This is obtainable by adjusting the phase of a backward-decaying solution of Eq. (19) so that, after forward numerical integration of Eqs. (18), γ does $\to \pi/2$ as $\tau \to \infty$ (see Fig. 2).

Since Eq. (18) are invariant under a time reversal, the transition pattern is the same (except for the scaling values $k_1 k_2$) at all junctions with the singular arc, whether arriving or departing. Since the resulting $d\gamma/d\tau$ is of order 1, the optimal lift, proportional of course to $\dot{\gamma}$, is of the order $\epsilon^{-1/2}$, rather than ϵ^{-1} as would be the case for maxL/D.

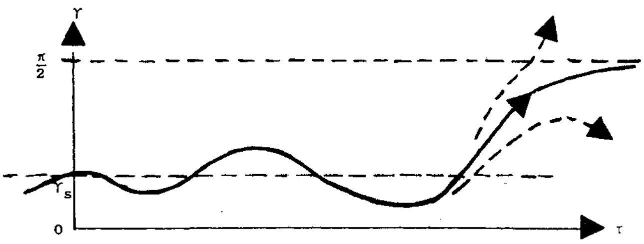


Fig. 2 γ-history during transition from singular arc.

A similar analysis applies to the transition in Fig. 1c from $h=0$ to the singular arc: the appropriate solution of Eq. (1) must have $d\gamma/d\tau=0$ at $\gamma=0$, and is found by adjusting the phase of a forward damped solution of Eq. (3) so that, after backward numerical integration of Eq. (1), $d\gamma/d\tau$ does vanish simultaneously with γ.

It is interesting to compare the damped oscillation about the singular arc with the optimal pattern described by Dixon[4] for a different model in which induced drag was ignored, but bounds on $|L|$, and hence on $|\dot{\gamma}|$, were introduced. The singular arc of Fig. 1 now was of second order, since $\dot{\gamma}$ became the control, and junction with the singular arc had to be accomplished by switching back and forth, with ever increasing frequency, between maximum positive and negative lift.

III. The Time-Loss

Now we will evaluate the time-loss, as compared with the ideal cornered trajectories of Fig. 1 when $\epsilon=0$, during a transition described by Eqs. (18). Since, from Eqs. (1) and (2),

$$t=m\int\frac{V\mathrm{d}V+g\mathrm{d}h}{V(T-D_0-\epsilon^2L^2/D_0)} \tag{21}$$

the dominant time-loss is the sum of two parts; 1) the direct increase in time on the transition path due to the induced drag ϵ^2L^2/D_0, and 2) the time-loss when $\epsilon=0$, caused by following the transition path instead of the cornered path. The first part is

$$\Delta_l t=\frac{\epsilon^2}{(T-D_0)D_0}\int L^2\mathrm{d}t=\frac{k_l^{1/4}\epsilon^{3/2}}{2k_2^{3/4}}\int_{-\infty}^{\infty}\left(\frac{\mathrm{d}\gamma}{\mathrm{d}\tau}\right)^2\mathrm{d}\tau \tag{22}$$

The second part may be evaluated by Green's Theorem:

$$\Delta_2 t=m\Big(\int_{\substack{\text{transition}\\ \text{path}}}-\int_{\substack{\text{cornered}\\ \text{path}}}\Big)\frac{V\mathrm{d}V+g\mathrm{d}h}{V/T-D_0)}$$

$$=m\int\int\left\{\frac{\partial}{\partial V}\left[\frac{g}{V(T-D_0)}\right]-\frac{\partial}{\partial h}\left(\frac{1}{T-D_0}\right)\right\}\mathrm{d}V\mathrm{d}h$$

$$=m\int\int\frac{g}{V^2(T-D_0)^2}F(h,V)\mathrm{d}V\mathrm{d}h \tag{23}$$

Figure 3 shows the appropriate area in the case of transitions from the singular arc to a vertical climb $\gamma=+\pi/2$. The shaded areas each contribute positively to $\Delta_2 t$, since the line integrals are taken clockwise or counterclockwise, according to $F<0$ or $F>0$ (area to right or to left of the singular arc).

The element of area $\mathrm{d}V\mathrm{d}h$ may be replaced by

$$\mathrm{d}A_l\mathrm{d}B_l/J(A_l,B_l/V,h)$$

where $A_l=-F$, and

$$B_l=\int_t^{\infty}(1-\sin\gamma)\mathrm{d}t$$

Table 1 Transition to $\gamma=\pi/2$

γ_S(rad)	$(2/\epsilon^{3/2})(k_2^{3/4}/k_l^{1/4})(\Delta_l t+\Delta_2 t)$	$\max\left\lvert\frac{\mathrm{d}\gamma}{\mathrm{d}\tau}\right\rvert$
– .2[a]	1.09	0.63
0 0	0.83	0.53
0.2	0.61	0.46
0.4	0.37	0.42
0.6	0.28	0.27
0.8	0.20	0.15

[a] Junction of $-\gamma_S$ with $\gamma=\pi/2$ is identical to junction of $+\gamma_S$ with $\gamma=-\pi/2$.

$\dot{A}_l$ and $\dot{B}_l$ are easily expressed linearly in terms of $\dot{V}$ and $\dot{h}$; the Jacobean is found to be

$$J\left(\frac{A_l,B_l}{V,h}\right)=\frac{m(gF_V-VF_h)}{V(T-D_0)}(1-\sin\gamma_S)$$

Hence,

$$\Delta_2 t=\frac{g}{V(T-D_0)(gF_V-VF_h)}\int_{-\infty}^{\infty}\frac{A_l^2}{2}\mathrm{d}B_l$$

$$=\frac{1}{2k_l(1-\sin\gamma_S)}\int_{-\infty}^{\infty}\dot{S}^2(1-\sin\gamma)\mathrm{d}t$$

Therefore

$$\Delta_2 t=\frac{k_l^{1/4}\epsilon^{3/2}}{2k_2^{3/4}}\int_{-\infty}^{\infty}\frac{1-\sin\gamma}{1-\sin\gamma_S}\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\tau}\right)^2\mathrm{d}\tau \tag{24}$$

For junctions with arcs $\gamma=-\pi/2$, a similar analysis leads to the same expression for $\Delta_l t$, while the factor $(1-\sin\gamma)/(1-\sin\gamma_S)$ in the integrand for $\Delta_2 t$ must replaced by $(1+\sin\gamma)/(1+\sin\gamma_S)$. For the junction with the state-constraint $h=0$ of Fig. 1c, this factor is replaced by $\sin\gamma/\sin\gamma_S$.

In every case the two time-losses $\Delta_l t$ and $\Delta_2 t$ are in the ratio 3:1, since rescaling of the independent variable of Eqs. (8) by a factor R leads to $\Delta_l t\Rightarrow\Delta_l t/R, \Delta_2 t\Rightarrow R^3\ \Delta_2 t$. Since $\Delta_l t+\Delta_2 t$ must be minimized at $R=1$, the 3:1 ratio is established. This ratio has been verified numerically for a variety of values of γ_S, along with the iterative computation of the appropriate solution of Eqs. (8). Table 1 shows a dimensionless time-loss as well as the maximum $|\mathrm{d}\gamma/\mathrm{d}\tau|$ for a few values of γ_S.

IV. Other Transitions

Consider now the transition between the vertical dive $\gamma=-\pi/2$ with the state-constraint $h=0$ of Fig. 1c. Along the constraint, λ_h is augmented by a negative quantity $\mu(t)$ such that h remains zero. In the limiting case $\epsilon=0$, this requires that $\sin\gamma=0$ maximize $\mathcal{H}$ (with $\lambda_\gamma=0$), and hence that S remain zero along the constraint, and $\lambda_V=1/\dot{V}$. However, $\dot{S}$ assumes the positive value $-gF/V(T-D_0)$ immediately prior to arrival at the constraint. For $\epsilon>0$ it will turn out that S is a positive quantity of order $\epsilon^{2/3}$ on arrival at the constraint. In fact, $\ddot{\gamma}$ again is given essentially by

$$\ddot{\gamma}\cong-(k_2/\epsilon^2)S\cos\gamma$$

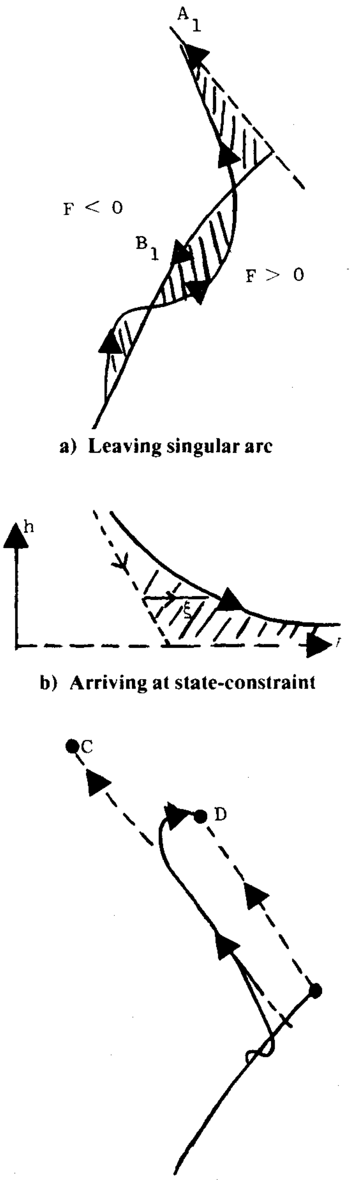
and now

$$\dot{S}\cong k_3=g|F|/[V(T-D_0)] \tag{25}$$

Introducing the rescaled variables

$$\tau=-[(k_2k_3)^{1/3}/\epsilon^{2/3}]t \tag{26a}$$

$$\sigma=k_2^{1/3}S/(k_3\epsilon)^{2/3} \tag{26b}$$



a) Leaving singular arc

b) Arriving at state-constraint

c) Final leveling off to $\gamma=0$

Fig. 3 Smoothed-out corners in $V-h$ plane.

the rescaled time now being measured backwards from the constraint, we obtain

$$\frac{d^2\gamma}{d\tau^2}=-\sigma\cos\gamma \tag{27a}$$

$$\frac{d\sigma}{d\tau}=-1 \tag{27b}$$

with $\gamma=d\gamma/d\tau=0$ at $\tau=0$ (since the optimal lift L has to be continuous at $\tau=0$), and $\gamma=-\pi/2$ as $\tau\to\infty$. Near $\gamma=-\pi/2$, τ is large, and

$$\frac{d^2(\pi/2+\gamma)}{d\tau^2}\cong\tau\left(\frac{\pi}{2}+\gamma\right)$$

The boundary condition $\gamma\to-\pi/2$ as $\tau\to\infty$ thus uniquely determines $\sigma(0)$; it has been found numerically to be 1.202702.

The time-loss again is made up of two parts:

$$\Delta_1 t=\frac{\epsilon^2}{D_0(T-D_0)}\int L^2 dt=\frac{k_3^{1/3}\epsilon^{4/3}}{2k_2^{1/3}}\int_0^\infty\left(\frac{d\gamma}{d\tau}\right)^2 d\tau \tag{28}$$

and

$$\Delta_2 t=\frac{mg|F|}{V^2(T-D_0)^2}\int\int dV dh$$

and the element of area $dVdh$ may be replaced by $d\xi dh$, where

$$\dot{\xi}=\dot{V}+\left(\frac{T-D_0}{mV}+\frac{g}{V}\right)\dot{h}=\frac{T-D_0}{m}(1+\sin\gamma)$$

Hence,

$$\Delta_2 t=\frac{k_3^{1/3}\epsilon^{4/3}}{k_2^{1/3}}\int_0^\infty[1+\sin\gamma(\tau)]\left(\int_0^{\tau'}[-\sin\gamma(\tau')]d\tau'\right)d\tau \tag{29}$$

In this case,

$$\Delta_1 t=2\Delta_2 t=0.404(k_3^{1/3}/k_2^{1/3})\epsilon^{4/3}$$

the 2:1 ratio being a consequence of the fact that rescaling of the independent variable of Eqs. (27) changes $\Delta_2 t$ into $R^2\Delta_2 t$.

Consider finally the case in which initial and/or final γ is prescribed, e.g., to be zero. A transition to $\gamma_f=0$ is governed by

$$\ddot{\gamma}=-(k_4/2\epsilon^2)\cos\gamma \tag{30}$$

where

$$k_4=SD_0/mV^2\lambda_V \tag{31}$$

where S is now positive, so that λ_V is no longer equal to $m/(T-D_0)$. λ_V and λ_h, and hence S, are determined, using $\epsilon=0$, from their rates, together with their values at junction with the singular arc (or with the state-constraint).

Introducing

$$\tau=(\sqrt{k_4}/\epsilon)t \tag{32}$$

we obtain

$$\frac{d^2\gamma}{d\tau^2}=-\tfrac{1}{2}\cos\gamma \tag{33}$$

and hence

$$\frac{d\gamma}{d\tau}=-\sqrt{1-\sin\gamma} \tag{34}$$

and

$$\tau_f-\tau=\sqrt{2}\ln\frac{\tan(\pi/8)}{\tan(\pi/8-\gamma/4)} \tag{35}$$

The time-loss is

$$\Delta t=[\lambda_V(V_C-V_D)+\lambda_h(h_C-h_D)]_{\epsilon=0}$$

D being the final position, and C the position that would have been reached at the final time if γ had been maintained at $\pi/2$. Note that the path leaves the singular arc earlier because of the end-constraint $\gamma_f=0$. Thus,

$$\Delta t=\frac{\lambda_V}{V}(E_C-E_D)+\frac{S}{V}(h_C-h_D) \tag{36}$$

E_c, E_D denoting the values of the energy state at C and D. But

$$E_C-E_D=\int\frac{\epsilon^2 L^2}{D_0 m}V dt=\epsilon\sqrt{k_4}\frac{mV^3}{D_0}\int_0^{\pi/2}\sqrt{1-\sin\gamma}\,d\gamma \tag{37}$$

and

$$h_C - h_D = \int V(1-\sin\gamma)\,dt = \frac{\epsilon V}{\sqrt{k_4}} \int_0^{\pi/2} \sqrt{1-\sin\gamma}\,d\gamma \qquad (38)$$

Thus

$$\Delta t = \frac{\epsilon}{\sqrt{k_4}} \left(\frac{mV^2\lambda_V k_4}{D_0} + S\right) 2(\sqrt{2}-1)$$

i.e.,

$$\Delta t = (\epsilon S/\sqrt{k_4})\,4(\sqrt{2}-1) \qquad (39)$$

A similar analysis would apply to a transition from a prescribed $\gamma_0 = 0$ to a vertical dive $\gamma = -\pi/2$; here $S<0$, $\tau = (\sqrt{k_4}/\epsilon)t$, with $k_4 = |S| D_0 / mV^2\lambda_V$, leading to

$$\tau = \sqrt{2}\,\ell n \frac{\tan(\pi/8)}{\tan(\pi/8+\gamma/4)}$$

and

$$\Delta t = (\epsilon |S| / \sqrt{k_4})\,4(\sqrt{2}-1)$$

V. Range of Validity of the Analysis

The most gradual transitions have been shown to be those to or from the singular arc, the unit of transition time ($\tau = 1$) being here of order $\epsilon^{1/2}$ rather than $\epsilon^{2/3}$ or ϵ. The corresponding time-loss is of order $\epsilon^{3/2}$ rather than $\epsilon^{4/3}$ or ϵ. Note that, if, as is customary, induced drag with $L = mg$ is included along the singular arc in $D_0(h, V)$, the resulting time-correction is only of order ϵ^2. From the point of view of this analysis, this inclusion is thus optimal. To be sure, the entire analysis is more meaningful for an airplane with very high L/D than for a typical supersonic jet airplane, especially since turn-rates at high altitudes become severely limited.

In order to check the validity of the underlying assumption that γ changes more rapidly than h and V, we may compute the time

$$t_I = \sqrt{\epsilon}/(k_1 k_2)^{1/4} \qquad (40)$$

corresponding to $\tau = 1$ in transitions to or from the singular arc, for the special case

$$\left.\begin{aligned} &T = \text{const} \\ &D = KV^2 e^{-h/H} \end{aligned}\right\} \qquad (41)$$

Here

$$F = KV^2(3 + V^2/gH)e^{h/H} - T \qquad (42)$$

and, for the singular arc,

$$e^{h/H} = (KgH/T)\nu^2(3+\nu^2)$$

and

$$\sin\gamma_S = \frac{T}{W} \frac{(2+\nu^2)(3+2\nu^2)}{(3+\nu^2)(3+(7/2)\nu^2+(3/2)\nu^4)} \qquad (43)$$

where

$$\nu = V/\sqrt{gH} \qquad (44)$$

If we take H to be 8.5 km, $\sqrt{gh}$ is approximately 290 m/sec, close to the speed of sound.

The transition time-unit is

$$t_I = \sqrt{\epsilon}\left(\frac{W}{T}\right)^{1/2} \sqrt{\frac{H}{g}} \frac{\nu\sqrt{3+\nu^2}}{(3+(9/2)\nu^2+\frac{1}{2}\nu^4)^{1/4}} \qquad (45)$$

But

$$\frac{d\nu}{dt} = \left(\frac{T}{W}\right)\sqrt{\frac{g}{H}} \frac{\nu^2(2+\nu^2)}{(1+\nu^2)(6+\nu^2)} \qquad (46)$$

and

$$\frac{dh}{H} = 2\frac{d\nu}{\nu}\left(1+\frac{\nu^2}{3+\nu^2}\right) \qquad (47)$$

The change in h/H and the relative change in V during transition, i.e., during a time t_I, is small, provided that

$$\sqrt{\epsilon}\left(\frac{T}{W}\right)^{1/2} \frac{2\nu^2(3+2\nu^2)(2+\nu^2)}{\sqrt{3+\nu^2}(1+\nu^2)(6+\nu^2)(3+(9/2)\nu^2+\frac{1}{2}\nu^4)^{1/4}} \ll 1 \qquad (48)$$

As expected, this condition is met at low subsonic speeds, e.g., $\nu \leq 0.5$, especially for a low-powered airplane with high L/D, but the analysis becomes questionable for high-powered aircraft at supersonic speeds.

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